The why, what and how of working with exemplification in mathematics teacher education[1]

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Summary

This paper focuses on working deliberately with secondary mathematics teachers on exemplification. The context is the Wits Maths Connect Secondary (WMCS) longitudinal research and professional development project in South Africa, where exemplification, specifically example sets informed by principles of variation, is a key focus in our work with teachers. We describe *why* we have come to focus on exemplification. We illustrate the *what and how* of this work through a selection of tasks where teachers are offered opportunity to learn about variance amidst invariance in example sets and what they afford.

Key Words: mathematics, teacher professional development, exemplification, secondary

Introduction

Theoretical and empirically supported arguments have been made for the centrality of examples in mathematics teaching and learning (e.g. Bills & Watson, 2008; Zaslavsky, online first) and how such learning is made possible by 'careful and knowledgeable' use of examples in mathematics teaching (Watson & Chick, 2011), and in mathematics teacher education (e.g. Peled & Balacheff, 2011). Building on this domain of work, this paper focuses on the *why, what and how* of working deliberately with secondary mathematics teachers on their use of examples in their teaching, or what we refer to more simply as *exemplification*. The context of our work is the Wits Maths Connect Secondary (WMCS) project where *exemplification*, and more specifically example sets informed by principles of variation, is a key element of our professional development work with teachers. We distinguish between the *modelling* of exemplification and *mediating* exemplification with teachers. Modelling takes place in our mathematics-focused professional development work. Mediation, on the other hand, is central to the teaching-focused

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[1] An substantially extended version of this paper will appear in the forthcoming *Handbook on Mathematics Teacher Education, Volume 1.*

aspects of our work. We will illustrate this distinction as we describe opportunities for teachers to learn about exemplification, and how the notion of variation provides a means for teachers to construct, critique and revise examples sets for use with their learners[2].

Why exemplification? The WMCS theory of teaching and initial research

The WMCS is a longitudinal research-linked professional development project aimed at improving mathematics teaching in socio-economically disadvantaged schools in one province in South Africa. Our work is shaped, on the one hand, by on-the-ground realities of mathematics teaching and learning, and on the other, by an orientation to the activity of teaching as 'social'. We draw from key tenets of sociocultural theory, where *mathematics* is viewed as an inter-connected network of scientific concepts, and *mathematics teaching* therefore as geared towards the mediation and appropriation of the increasingly sophisticated and increasingly general ways of thinking that constitute progression in the discipline (Vygotsky, 1978). Teaching as an activity is not only goal-directed but also always about *something* (Alexander, 2000). Bringing what students are to know and be able to do into focus – its mediation - is the teacher's work. We call this 'something' the *object of learning*. In practical terms, it is akin to a lesson goal, but worded so that the mathematics of the goal is made clear. In line with previous research, mediational means are understood as cultural tools and/or resources in the practice of teaching (Adler, 2001).

Traditional forms of teaching are common across the world (Nachlieli & Tabach, online first), and unsurprisingly were the dominant forms observed in initial observations in our project schools and classrooms. Our analysis of video-records of lessons showed that they were characteristically incoherent. While teachers were following high levels of curriculum prescription, and learners were attentive and 'working', the intended mathematical message in a lesson was often not clear leading us to wonder how specific mathematical goals influenced lesson development activity for teachers. In our terms, the mathematical object of learning was out of focus. There was no apparent mathematical 'story' linking what learners were to know and be able to do. For example, in a four-part lesson ostensibly on multiplying algebraic expressions, each part offered a different rule, thus presenting an incoherent and fragmented notion of the products (Adler & Venkat, 2014).

In the context described above, and a principle that good professional development begins with what teachers bring and so who and where they are, it made sense that our professional development work should attend to strengthening teachers' exemplification and the quality of their explanations. From a Vygostkian perspective, examples are *symbolic mediators* of mathematics. Symbolic mediators include different signs, symbols, writing, formulae and graphic organisers – all possible elements of mathematical examples. As Kozulin (2003) explains, one cannot take for granted that learners will detect symbolic relations, no matter how obvious they might seem to the teacher.

> Symbols may remain useless unless their meaning as cognitive tools is properly mediated … the mere availability of signs or texts does not imply that they will be used by students as psychological tools … (Kozulin, 2003 p.24)

The implications for teaching and learning follow. Appropriation of psychological tools and more connected scientific mathematical concepts requires deliberate teaching of symbolic tools. This includes their systematic organisation and an emphasis on their generality and

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[2] We use the terms learner/s and student/s interchangeably.

application.

Symbolic tools (e.g. letters, codes, mathematical signs) have no meaning whatsoever outside the cultural convention that infuses them with meaning and purpose. (Kozulin, op cit p.26).

Indeed, instructional examples are just such cultural conventions. Examples and exemplification thus form part of a framework we have developed over time that informs both our research and development work. This framing, named *Mathematical Discourse in Instruction* (MDI), enables us to describe and analyse what it is teachers do, and then to work with them developmentally on the mathematical quality of their teaching.
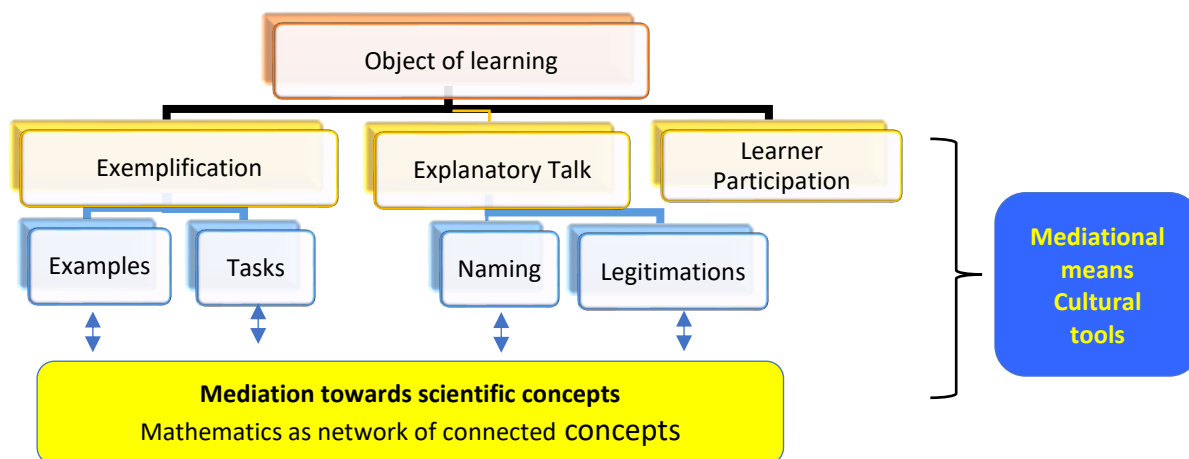

Fig.1: Constitutive elements of MDI (Adapted from Adler & Ronda, 2015)

As represented in figure 1, MDI focuses on four key elements of mathematics teaching, with *a lesson* as our *unit of analysis*: the *object of learning*, or lesson goal, and three mediational means, or cultural tools, *exemplification*, *explanatory talk* and *learner participation*. The *object of learning* in any lesson could be a concept, a procedure or mathematical practice, together with the relevant capability. This leads to *exemplification* and more specifically to examples and associated tasks that can be used to bring the object of learning into focus with learners. With respect to examples, we are interested in their selection and sequencing and how these accumulate within and across lessons. We draw on the work of Watson & Mason (2006), who in turn draw on Marton & Tsui (2004), to describe key features of a mathematical object and/or movement towards generality across a sequence of examples. An example set that brings attention to *similarity* across examples, and so to that which is invariant, offers opportunity to identify key features and/or to generalise. If a set of examples brings attention to *contrast*, and so to what something *is* in relation to what it *is not*, or to a different class, opportunities are made available to recognise boundaries between classes of examples. This provides further opportunity to generalise but not overgeneralise. When two examples that are carefully varied are *juxtaposed*, they can draw quite specific attention to a key feature of an object.

We note the third and fourth elements of MDI, though they are not in focus here. *Explanatory communication,* includes attention to naming/word use (what is said and what is written) and substantiations of mathematics as specialised knowledge (what counts as mathematical knowledge). *Learner participation* focuses on what learners do and say with regards to the mathematics they are learning. We consider whether and how learner talk moves

beyond the very prevalent but limited offering of single words in chorus, to responding to and asking questions, and to more open dialogue with the teacher and/or other learners.

The overarching importance of the framework is the emphasis on the coherence of a lesson and thus how all elements interact, how they link back to the object of learning and open opportunities to learn mathematics. The constitutive elements of MDI apply to any mathematics classroom, whatever the pedagogy.

Why exemplification? Mathematics education research

Interest in exemplification as a research field in mathematics education goes back many years. For example, Mason and Pimm (1984) explored the role of generic examples in mathematics education. Dahlberg and Housman (1997) investigated the generation of examples in learning advanced mathematics. It is fair to say that interest in examples in mathematics and mathematics education follows from a basic maxim that initial experiences of mathematical concepts and procedures, given their abstract nature, will be through some exemplification: through examples and the tasks in which they are embedded. Goldenberg & Mason (2008) described examples as cultural mediating tools, and linked examples with the notion of variation:

> Examples can … usefully be seen as *cultural mediating tools* between learners and mathematical concepts, theorems, and techniques. They are a major means for 'making contact' with abstract ideas and a major means of mathematical communication, whether 'with oneself', or with others. Examples can also provide context, while *the variation in examples* can help learners distinguish essential from incidental features and, if well selected, the range over which that variation is permitted. (Goldenberg & Mason, 2008 p. 184, *our emphasis*).

The resonance with our theoretical orientation to examples as symbolic tools, and to our work on and with exemplification in the WMCS project is apparent.

A 2006 PME Research Forum on exemplification culminated in a special issue of *Educational Studies in Mathematics* in 2008 (Bills & Watson 2008) and was a catalyst for a follow-on conference focused on the role of examples in argumentation and proof, and a related special issue of the *Journal of Mathematical Behavior* (JMB) in 2011 (Antonini, Presmeg, Mariotti, & Zaslavsky, 2011). Both issues provide reviews of research in the field. Here we zoom in on the papers focused on the role of examples in *teachers' learning of mathematics in teacher educatio*n, or *teachers' use and awareness of examples in their te*aching. In this way, the crucial role in teaching of choosing and using (instructional) examples becomes evident. The value of a deliberate focus on exemplification in mathematics teacher education follows.

Studies of the forms and functions of teachers' example-use have extended to both elementary and secondary mathematics teaching, and to pre-service and experienced teachers. Rowland (2008) explored example-use across 24 lessons taught by pre-service elementary teachers. He identified four categories of example-use: variability, sequencing, representations, and lesson objectives. These analytic distinctions in turn provided insight into aspects of teachers' mathematical knowledge-in-use in teaching: that this entails variation across a set of examples, their sequencing and link with lesson goals. Each of these aspects features in MDI.

Using their own experience of example-use when working on a task about polynomial functions, and example-use in the lessons of an experienced secondary teacher teaching decimals, fractions and percentages, Watson & Chick (2011) reinforce "how careful and

knowledgeable teachers need to be" to bring about "alignment between the learners' engagement and teacher's intentions" (p.294). Put differently, the affordances of an example or an example set are not self-evident to learners. It takes a knowledgeable teacher, with some fluency in example-use to draw learners' attention to, and engage them with what is significant for their mathematics learning.

In an earlier study of example-use by experienced secondary teachers, Zodik & Zaslavsky's (2008) illuminated that teachers were not necessarily aware of their example-use and related rationales. From their observations and analysis of teaching, they distinguished between teachers' pre-planned use of examples, and their spontaneous use as these arose in the course of teaching. They also revealed that example choices can either facilitate or impede students' learning, and consequently the choice of examples in teachers' work is not trivial. They went on to lament the lack of deliberate attention to exemplification in mathematics teacher education.

> … numerous mathematics teacher education programmes do not explicitly address this issue and do not systematically prepare prospective teachers to deal with the choice and use of instructional examples in an educated way. (p.166)

Zodik & Zaslavsky have argued further for its place as part of specialised knowledge for teaching (Ball, Thames, & Phelps, 2008):

> The knowledge teachers need for meeting the challenge of judiciously constructing and selecting mathematical examples is a special kind of knowledge. It can be seen as core knowledge needed for teaching mathematics. … engaging teachers in generating or choosing instructional examples can be a driving force for enhancing these elements of their knowledge (Zodik & Zaslavsky, 2009).

Zazkis & Leikin's (2008) study of the role of examples in defining and definitions also argues for the development of this specialized content knowledge for teaching mathematics. They presented a group of 40 pre-service secondary teachers with the task of giving "as many examples as possible for a definition of a square" (p. 134). They were interested in the prospective teachers' concepts of a square in the first instance, and then their meta-mathematical concept of a definition. An additional research question related to the usefulness of a three-dimensional framework for analysing examples: accessibility and correctness, richness, and generality. The student teachers' definitions and related example-use generated a large number of examples of definitions, including more and less rigorous definitions, as well as some incorrect ones. Of interest to us in Zazkis & Leikin's study was the follow-on task given to the same group of prospective teachers later in the year. They presented the prospective teachers with 24 examples of definitions, most of which were selected from their teacher-generated examples of definitions and with additions of some produced by 'experts'. The task for the prospective teachers was to evaluate the validity of each of the definitions provided. The discussion amongst the teachers about the validity of the various definitions revealed movement between mathematical validations and pedagogical ones. Some teachers evaluated validity in terms of what they thought would be appropriate for school teachers, without attention to related mathematical rigour. Zazkis & Leikin concluded by suggesting that the tasks offered "a valuable activity, both mathematical and pedagogical, to promote a deeper conceptual understanding of mathematics in general and of the nature and role of definitions in particular" (p.147). The critical point here is that these tasks in a teacher education setting were mathematical and meta-mathematical, intended to strengthen teachers' knowledge of examples of definitions and

defining. The tasks were not designed for explicit attention to choosing and using instructional examples. The value of this study is that it points to the value of such activities in mathematics teacher education, and the specialized knowledge these can lever up for teachers.

There is thus a considerable literature foregrounding the significance exemplification as specialised knowledge for teaching. This leads to the question of what and how this is attended to in mathematics teacher education, and hence the focus of this paper. Such attention, we will argue can be fostered through the use of variation. The importance of variation in mathematics teaching has a long history, dating back to Dienes' (1960) work on perceptual variation. In mathematics education variation has come back into focus in more recent years through the work of Watson & Mason (2006), particularly through their vivid illustration of variance amidst invariance as a tool for engaging with generality and with mathematical structure, and through carefully structured example sets.

More generally, application of Variation Theory (Marton & Tsui, 2004) to mathematics education has ranged from studies of variation in textbook example sets (e.g. Sun, 2011) to teachers' learning through lesson study informed by variation theory (e.g. Runesson, 2008) and to learners' learning (e.g. Kullberg, Runesson Kempe & Marton, 2017). Kullberg et al (op cit) emphasise how attention to variation can enable critical features of the object of learning to come into focus. They make the important observation that multiple examples are not simply cumulative. The ordering of examples, their simultaneous presentation, and the teacher drawing attention to similarities and differences are critical. We agree, and we have argued previously that research related to an entire lesson needs to attend to the accumulating example space.

> The research on examples, [however,] while illuminating of what teachers do and why, does not enable a view of whether and how examples accumulate to bring the object of learning into focus for learners, and whether there is movement towards generality (Adler & Ronda, 2017, p. 68).

In Variation Theory terms, learners' attention needs to be drawn to key features of a mathematical object such as aspects of mathematical structure the teacher wishes to make visible. This is a function of a set of examples, how they are organized to illuminate variance amidst invariance and thus possibilities made available for generalizing, and/or recognizing structure or key features.

Of course, examples are always embedded in a task. Thus while examples are selected as particular instances of the general case in focus, and for drawing attention to relevant features, generality and/or structure, tasks are designed to bring particular capabilities to the fore (Marton & Pang, 2006). For example, expanding $a(b + c)$ and factoring $ab + ac$ are different tasks. Different tasks require different actions, at different levels of complexity, and so make available different opportunities for mathematics learning. In our work, we link examples and tasks in our consideration of exemplification, since an example or an example set is always embedded in a task. Indeed, it is this that makes an example 'instructional'.

Exemplification: the what and how

As we noted earlier, the MDI framework informs our teaching of our mathematics-for-teaching course called Transition Maths 1. We focus on the course since it is the major context in which we deliberately "teach" exemplifying/exemplification as a key mathematics teaching practice. TM1 is structured so that approximately two-thirds of the time teachers focus on their

own learning of mathematics. In these mathematics sessions, we *model* exemplification as a mathematics teaching practice. Teachers work on mathematical tasks where the objects of learning are key mathematical concepts, procedures and/or practices, the selection of which has been influenced by the South African school curriculum. In general, these tasks and activities provide opportunities to revisit and deepen their knowledge of the mathematics they teach (Zazkis, 2011), as well as activities that extend their knowledge of school mathematics. From this activity they build generality, focus on mathematical structure and engage with mathematical procedures and their rationales. The tasks and example sets offered to teachers are carefully selected to model and illustrate the forms of variation described earlier. For example, the task in figure 2 deals with informal methods of finding solutions to quadratic equations in factorised form. We ask teachers to find numerical values without using formal procedures. We also ask them to reflect on "what changes" and "what stays the same" and to consider the impact of this variation on how they approached each example. All this work has a mathematical focus.

Give values for x to make the statements true*:*

   a) $x(x - 2) = 8$

   b) $x(x - 2) = 0$

   c) $x(x - 2) = x$

   d) $(x - 1)(x + 2) = 4$

   e) $(x - 1)(x + 2) = 0$

Fig. 2: Mathematics tasks for quadratic equations

While the course presenter frequently points to the variance amidst invariance in the example sets, this is to mediate the mathematics in play and only an implicit form of drawing attention to exemplification with variation. It is in the teaching sessions that we deliberately *mediate exemplification* as a teaching practice.

The remaining one-third of the course, and our particular interest in this paper, focuses on mathematics teaching. These teaching-focused sessions are structured to mediate all the components of the MDI framework. We work from the assumption that better teaching is characterised by more thoughtful selections of examples and tasks, and by mathematical explanations that focus explicitly on the mathematics the teacher intends the learners to learn. With respect to exemplification, we work with teachers on articulating the mathematical goals for a lesson (objects of learning), and then on choosing and using examples. Using principles of variation, we examine sets of examples that either we have constructed, or are available in textbooks or in a prescribed lesson plan, to ascertain what is possible to come into focus. A key strength here is that our focus is on issues that are sufficiently close to teachers' current practice, and to curriculum demands, as to be possible to implement. In the remainder of this paper, we elaborate our attention to exemplification as a key focus in our work and specifically how we mediate this with teachers.

Mediating exemplification in mathematics teacher education

We provide two illustrative cases of how we work with exemplification using variation in professional development. The cases involve algebra and function both of which are given substantial attention in the course. We distinguish between the learner task and the teacher

education task, and hence make explicit what it is about exemplification that we intend teachers to learn. Case 1 illustrates how we introduce teachers to ideas of variation in an example set. Case 2 extends ideas of variation in an example set to focus on connections between representations, leading to generalisation. Our work with teachers extends beyond this to having teachers apply ideas of variation to produce a new example set, as well as reflect and critique such, but space restrictions preclude a third case here. In the presentation, I will include a third case to illustrate more adequately, the progression in our mediation of exemplification.

Case 1 – Introducing teachers to variation in an example set to address learner error

Our first case has its roots in lesson study work with teachers (see Adler & Alshwaikh, in press) who had already completed the course and so had been introduced to variation. Case 1 connects directly into teachers' practices in two ways: (1) it deals with a prevalent and persistent error in the application of the distributive law and the use of brackets; and (2) it deals with meaning of algebraic forms. We have drawn on this very specific problem of practice to construct a learning opportunity in teacher education for the introduction of ideas of variation.

The **learner task** is framed by the following object of learning: "learners must be able to simplify expressions with brackets that appear in different positions" and contains the example set in figure 3. Teachers would typically ask learners to attempt the task individually and may then invite learners to work in pairs to compare their answers. Thereafter the answers might be discussed in a whole-class setting. The teacher would then draw attention to what is the same and different about each of the expressions and so the application of the distributive law.

Learner task

Simplify the following expressions:

a) $x + 3(x + 5)$
b) $(x + 3)x + 5$
c) $x - 3(x + 5)$
d) $(x + 3)(x + 5)$
e) $(x + 3) - (x + 5)$

Fig. 3: Learner example set involving application of the distributive law

Invariance here lies in the selection and order of symbols (numeric and algebraic). Variance is introduced in how the symbols are combined through operations and the position of brackets.

In the **teacher education task,** the five examples are set up as a collection of pairs of expressions, numbered 1-8, and with answers provided for convenience (see figure 4). These pairs are carefully juxtaposed to focus on particular learner errors. In this way teachers are invited to compare the following pairs from the learner task: (a)-(b), (a)-(c), (a)-(d) and (d)-(e). In comparing (a) and (b), we address the common error where learners do not consider a letter to the right of the bracket to be an instance of the distributive law. By contrast, in comparing (d) and (e) we address the overgeneralisation "brackets mean multiply".

Teacher task

Look at each pair of expressions:

1) $x + 3(x + 5) = 4x + 15$
2) $(x + 3)x + 5 = x^2 + 3x + 5$

3) $x + 3(x + 5) = 4x + 15$
4) $x - 3(x + 5) = -2x - 15$

5) $x + 3(x + 5) = 4x + 15$
6) $(x + 3)(x + 5) = x^2 + 8x + 15$

7) $(x + 3)(x + 5) = x^2 + 8x + 15$
8) $(x + 3) - (x + 5) = -2$

What varies?

What is invariant?

What mathematics is possible to learn through the variation?

Fig. 4: Teacher example set for application of the distributive law

When the task is presented as pairs of expressions, the contrast is more explicit because of the juxtaposition of items with minor visual differences. We ask teachers three key questions which we ask of all example sets: "what varies?", "what is invariant"? and "what mathematics is possible to learn from this variation?" This teacher education task provides several learning opportunities for teachers. Firstly, teachers see that working with variance amidst invariance provides a teaching strategy for choosing and/or designing example sets to focus on particular learner errors. Secondly, by focusing on pairs of examples, it is easier for teachers to identify variance amidst invariance. This shows how juxtaposition with minor variation has the potential to bring the object of learning more clearly into focus than it might in the original learner task.

Having identified variance amidst invariance, the next step is for teachers to produce a similar pairing and then a full example set related to the distributive law. The extent to which teachers can do this successfully gives us some insight into the sense they have made of this introduction to the principles of variation as well as their grasp of the structural aspects of the mathematics in focus. In figure 5 we illustrate two typical pairings that teachers propose:

$x + 3(x + 5) = 4x + 15$	$x + 3(x + 5) = 4x + 15$
$(x + 3x) + 5 = 4x + 5$	$x(+3x) + 5 = 3x^2 + 5$
Fig. 5a	*Fig. 5b*

Fig. 5: Teachers' extensions of the given example set

In figure 5a, the pairing draws attention to the matter of "do the brackets first". This new addition will provide the only instance in the example set where the bracket can first be simplified. Thereafter it is similar to example (2) in figure 4, where the 5 is then added. There is thus further potential for juxtaposition with another example in the set. In figure 5b the pairing draws attention to the distinction between sign and operation. The bracket in the new expression

shifts the meaning from "add 3" to "positive 3". This inevitably leads to some discussion about whether the new example maintains the focus on the distributive law or whether the focus on sign versus operation diverts attention away from the intended object of learning.

We have learnt that teachers are easily able to identify the surface features of the variation and to produce their own examples of variation. However, in doing so, they may lose focus on the object of learning. Consequently, their suggested changes may simply generate an expression that varies rather than maintaining focus on the intended object of learning. So we recognise this as part of the journey of learning to work with principles of variation.

Case 2: Extending ideas of variation in an example set to attend to connections between representations and to generalise

This case is similar to the first in that it is drawn and adapted from lesson study work (see Adler & Ronda, 2017), and also close to teachers' practice. Here the example set (figure 6) is constructed to lead learners to generalise the impact of parameters on the graph of the quadratic function $y = ax^2 + q$ and hence to make connections between the equation and the graph.

The **learner task** involved a card matching activity where learners were given an example set containing six equations and six graphs on separate cards. They were required to match each graph with an equation. As the lesson progressed, the teacher worked with learners to generalise the impact of a change in the sign of a and the value of q on the graph.
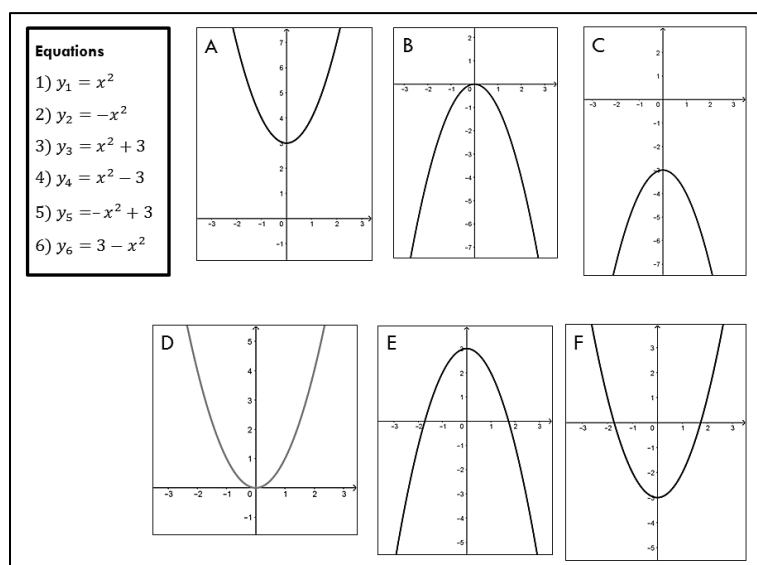

Fig. 6: Example set for learner task to match equations and graphs

In the session, teachers were first required to complete the card matching task. They spotted the "twist" designed into the original example set by the teacher in the lesson study: that equations 5 and 6 are the same and therefore there is no corresponding equation for graph C. Hence the learner task also involved producing an equation for graph C.

The **teacher education task** required teachers to compare the pairs of equations and graphs shown in figure 7, and to identify what varies, what is invariant and what mathematics is possible to learn through the variation.

Teacher task

Look at the following pairs of equations and graphs:

a) equation 1/graph D and equation 3/graph A
b) equation 1/graph D and equation 4/graph F
c) equation 1/graph D and equation 2/graph B

What varies?

What is invariant?

What mathematics is possible to learn through the variation?

Fig. 7: Teacher task for function matching task

As can be seen in the example set for the teacher education task, equation 1/graph D is held constant and the other member of the pair changes. This is intended to draw teachers' attention to how one might structure an investigation of the impact of two parameters across two different representations. Teachers were able to identify that in (a) and (b) the focus was the impact of q on the vertical position of the graph, and in (c) attention moves to the impact of the sign of a on the orientation of the graph. We then invited teachers to choose their own pair of equations/graphs to compare and to identify what mathematics was possible to learn from the pairing. As expected, a common pairing was equation 2/ graph B and equation 5/graph E which drew attention to the effect of q. We were encouraged that many groups also selected pairs that led to the following generalisations: "if a and q have the same sign, then the graph has no roots", and conversely, "if a and q have opposite signs, then the graph has two roots".

The use of juxtaposition in the teacher education task provides a structure for introducing variation in task design when more than one representation and more than one feature are in focus. In other words, the prompts for teachers provide a scaffold to think about which elements to attend to when designing future card matching tasks, and how to vary these features. At the same time, the design of the teacher education task suggests a possible teaching strategy for making the learner task more accessible to learners who may have difficulty in dealing with twelve different cards all at once.

Cases 1 and 2 provide some evidence of teachers' take up of the principles of variation in extending example sets during the course. As noted, we have additional cases that point to the range of issues that need attention when constructing a carefully designed and focused example set, for examplepaying attention to juxtaposition and what be generalised from the combination of pairs of examples. We are reminded here of Kullberg et al's warning that collections of examples are not necessarily cumulative.

Discussion and conclusion

Teaching, whatever its context and/or pedagogy, is purposive work. At the heart of this paper is how we work with teachers to develop their purposive and deliberate choice and use of examples in their teaching. We have argued both from the growing literature base and from our own research with teachers in the project, that focusing on exemplification in conjunction with principles of variation, and in particular attention to variance amidst invariance, is an important and necessary component of secondary mathematics professional development.

Through the cases above we have illustrated three important features of this work. Firstly, separating the learner and teacher education task is critical for being able to focus teachers' attention on what it is they are to be coming to know and be able to do: principles of variation and how to apply these to structure focused example sets. Secondly, the mathematical task for the learners needs to be familiar to teachers so that their attention is on the learning of exemplification. Thirdly, it is important to organise the teacher education tasks so that there is progression from becoming familiar with principles of variation at work in an example set, to being able to work with these when there are two (and possibly more) representational forms. As noted, applying these principles to constructing such sets will be illustrated in the presentation of this paper.

In conclusion, we reflect first on the nature of the tasks we use, and then on some of the challenges we have faced in our work. We have indicated how our tasks for mediating aspects of mathematics teaching remain close to teachers' 'predominantly traditional' practices. Much of the literature on exemplification tends to be related to teaching with rich tasks and inquiry-based pedagogies. We hope we have shown the importance of explicit attention to exemplification in relation to more traditional tasks focusing on key concepts and procedures in school mathematics. We have also hinted at point some challenges as teachers engage with exemplification informed by principles of variation. Teachers easily notice variation at a visual level such as changes in numbers, letters, orders of symbols, etc. While this is an important first step, it is insufficient to engage only with the visual features of an example in its particular representation. For example, example sets of algebraic equations do not immediately reveal the nature of their solutions. Drawing teachers' attention to variance amidst invariance is not trivial if it is to move them beyond superficial use of such in their teaching. We are cognisant that there is much still to explore in working with mathematics teachers on examples and example sets.

Acknowledgement

References

Adler, J. (2001). Teaching mathematics in multilingual classrooms. Dordrecht: Kluwer.

Adler, J., & Alshwaikh, J. (in press). A Case of Lesson Study in South Africa. In R. Huang, A. Takahashi & J. da Ponte (Eds.), Theory and practices of lesson study in mathematics: An international perspective. Dordrecht: Springer.

Adler, J., & Ronda, E. (2015). A framework for describing mathematics discourse in instruction and interpreting differences in teaching. African Journal of Research in Mathematics, Science and Technology Education. doi: 10.1080/10288457.2015.1089677

Adler, J., & Ronda, E. (2017). Mathematical discourse in instruction matters. In J. Adler & A. Sfard (Eds.), Research for educational change: Transforming researchers' insights into improvement in mathematics teaching and learning (pp. 64-81). Abingdon: Routledge

Adler, J., & Venkat, H. (2014). Teachers' mathematical discourse in instruction: Focus on examples and explanations. In H. Venkat, M. Rollnick, J. Loughran & M. Askew (Eds.), Exploring mathematics and science teachers' knowledge: Windows into teacher thinking (pp. 132-146). Abingdon, Oxon: Routledge.

Alexander, R. (2000). Culture and pedagogy: International comparisons in primary education. Oxford: Blackwell.

Antonini, S., Presmeg, N., Mariotti, M. A., & Zaslavsky, O. (2011). On examples in mathematical thinking and learning. ZDM: Mathematics Education, 43(2), 191-194. doi: doi:10.1007/s11858-011-0334-5

Ball, D., Thames, M., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? Journal of Teacher Education, 59, 389-407. doi: 10.1177/0022487108324554

Bills, L., & Watson, A. (2008). Editorial introduction, Editorial, Educational Studies in Mathematics, pp. 77-79. Retrieved from http://search.ebscohost.com/login.aspx?

direct=true&db=ehh&AN=34227504&site=ehost-live

Dahlberg, R. P., & Housman, D. L. (1997). Facilitating learning events through example generation. Educational Studies in Mathematics, 33, 283-299. doi: doi:10.1023/A:1002999415887

Dienes, Z. (1960). Building up mathematics. London: Hutchinson Educational Ltd.

Goldenberg, P., & Mason, J. (2008). Shedding light on and with example spaces. [Article]. Educational Studies in Mathematics, 69(2), 183-194. doi: 10.1007/s10649-008-9143-3

Kozulin, A. (2003). Psychological tools and mediated learning. In A. Kozulin, B. Gindis, V. Ageyev & s. Miller (Eds.), Vygotsky's educational theory in cultural context (pp. 15-38). New York: Cambridge University Press.

Kullberg, A., Runesson, K. & Marton, F. (2017). What is made possible to learn when using the variation theory of learning in teaching mathematics? ZDM Mathematics Education, 49, 559–569. DOI 10.1007/s11858-017-0858-4

Marton, F., & Pang, M. F. (2006). On some necessary conditions of learning. The Journal of the Learning Sciences, 15(2), 193–220.

Marton, F., & Tsui, A. B. M. (2004). Classroom discourse and the space of learning. New Jersey: Lawrence Erlbaum Associates.

Mason, J., & Pimm, D. (1984). Generic examples: Seeing the general in the particular. Educational Studies in Mathematics, 15(3), 277-289.

Nachlieli, T., & Tabach, M. (online first). Ritual-enabling opportunities to learn in mathematics classrooms. Educational Studies in Mathematics.

Peled, I., & Balacheff, N. (2011). Beyond realistic considerations: modeling conceptions and controls in task examples with simple word problems. ZDM Mathematics Education, 43, 307-315. doi: DOI 10.1007/s11858-011-0310-0

Pournara, C., Hodgen, J., Adler, J., & Pillay, V. (2015). Can improving teachers' knowledge of mathematics lead to gains in learners' attainment in mathematics? South African Journal of Education, 35(3), 1-10. doi: 10.15700/saje.v35n3a1083

Rowland, T. (2008). The purpose, design and use of examples in the teaching of elementary mathematics. Educational Studies in Mathematics, 69(2), 149-163.

Runesson, U. (2008). Learning to design for learning. In P. Sullivan & T. Wood (Eds.), Knowledge and beliefs in mathematics teaching and teacher development (pp. 153-172). Leiden: Sense.

Sun, X. (2011). "Variation problems" and their roles in the topic of fraction division in Chinese mathematics textbook examples. Educational Studies in Mathematics, 76(1), 65-85. doi:10.1007/s10649-010-9263-4

Vygotsky, L. S. (1978). Mind in society : the development of higher psychological processes. Cambridge: Harvard University Press.

Watson, A., & Chick, H. (2011). Qualities of examples in learning and teaching. ZDM, 43(2), 283-294. doi: 10.1007/s11858-010-0301-6

Watson, A., & Mason, J. (2005). Mathematics as a constructive activity: Learners generating examples. Mahwah, NJ: Lawrence Erlbaum.

Watson, A., & Mason, J. (2006). Seeing an exercise as a single mathematical object: Using variation to structure sense-making. Mathematical Thinking and Learning, 8(2), 91-111.

Zaslavsky, O. (online first). There is more to examples than meets the eye: Thinking with and through mathematical examples in different settings. The Journal of Mathematical Behavior.

Zazkis, R. (2011). Relearning mathematics: A challenge for prospective elementary school teachers. Charlotte, NC: Information Age Publishing

Zazkis, R., & Leikin, R. (2008). Exemplifying definitions: A case of a square. Educational Studies in Mathematics, 69(2), 131-148. doi: DOI: 10.1007/s10649-008-9131-7

Zodik, I., & Zaslavsky, O. (2008). Characteristics of teachers' choice of examples in and for the mathematics classroom. Educational Studies in Mathematics, 69(2), 165-182. doi: DOI: 10.1007/s10649-008-9140-6

Zodik, I., & Zaslavsky, O. (2009). Teachers' treatment of examples as learning opportunities. In M. Tzekaki, M. Kaldrimidou & C. Sakonidis (Eds.), Proceedings of the 33rd conference of the International Group of the Psychology of mathematics education (Vol. 5, pp. 425-432). Thessaloniki, Greece: PME.