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The Challenge of Proof in Abstract Algebra: Undergraduate Mathematics Students' Perceptions

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Summary

This study aims to investigate undergraduate Mathematics students' difficulties with mathematical proof in their first encounter with Abstract Algebra. Abstract Algebra, and Group Theory in particular, is one of the most demanding mathematical fields for novice students, due to its abstract nature. This difficulty appears to have an unfavorable consequence on the proof production. For the purposes of this qualitative study, there has been used the Theory of Commognition. Analysis suggests that proof is considered a difficult task mainly due to students' incomplete grasp of the newly introduced algebraic notions, and due to problematic application of the various proving strategies. This difficulty has often a negative impact on students' engagement with pure Mathematics in general, resulting novice undergraduate students' tendency to avoid the study of other Pure Mathematics modules.

Key Words: Proof, Abstract Algebra, commognition, university Mathematics Education

Introduction

Abstract Algebra is considered by the majority of undergraduate Mathematics students as one of the most demanding modules in their curriculum, in which they face both cognitive and metacognitive challenges (Ioannou, 2012). Often, after their first encounter with Abstract Algebra, students tend to avoid further modules in this area of Mathematics. Nardi (2000) attributes student difficulty with Abstract Algebra to its multi-level abstraction and the less-than-obvious, to novice students, raison d'être of concepts such as cosets, quotient groups etc. Dubinsky et al (1994, p268) have concluded that students after their first encounter with it, they avoid any further study, since it is "the first course in which students must go beyond 'imitative behavior patterns' for mimicking the solution of a large number of variations on a small number of themes."

In addition, Abstract Algebra requires "deeper levels of insight and sophistication" (Barbeau, 1995, p139). This challenge is also due to the fact that instructors, more often than not, do not give adequate time to students to reflect on the new material (Clark et al, 1997). Weber (2001, 2008) and Ioannou and Nardi (2010) suggest that cognitive difficulties are related to metacognitive issues concerning students' coping strategies in the learning process, for example proof production and consequently affective issues. Additionally, the students' introduction to the novel ideas of groups takes place in the unfamiliar academic context of large-scale lectures. This unfamiliarity is likely to exacerbate their difficulty with the topic (Mason, 2001). Also, as it is often suggested by research, lecturing to large student audiences has an arguable effect on student engagement.

Research on how Mathematics undergraduates cope with the complexity of university studies suggests that in proof production, an essential part of their studies in Mathematics and "the only means of assessing students' performance" (Weber, 2001, p101), students face difficulties of two categories: firstly, "they do not have an accurate conception of what constitutes a mathematical proof", and in addition they "may lack an understanding of a theorem or a concept and systematically misapply." (Weber, 2001, p102)

This study examines the difficulty students have with mathematical proof in the context of their first encounter with Abstract Algebra. Moreover, this study adopts a participationist perspective on learning, for the purposes of which there will be used the Commognitive Theoretical Framework (CTF) by Anna Sfard (2008). Presmeg (2016, p. 423) suggests, it is a theoretical framework of unrealised potential, designed to consider not only issues of teaching and learning of Mathematics per se, but to investigate "the entire fabric of human development and what it means to be human." It proves to be an astute tool for the comprehension of diverse aspects of mathematical learning, which although grounded on discrete foundational assumptions, can be integrated to give more holistic view of the students' learning experience (Sfard 2012).

Theoretical Framework

CTF is a coherent and rigorous theory for thinking about thinking, grounded on classical Discourse Analysis. It involves a number of different constructs such as metaphor, thinking, communication, and commognition, as a result of the link between interpersonal communication and cognitive processes (Sfard, 2008). In mathematical discourse, objects are discursive constructs and form part of the discourse. Mathematics is an *autopoietic system of discourse*, namely "a system that contains the objects of talk along with the talk itself and that grows incessantly 'from inside' when new objects are added one after another" (Sfard, 2008, p129). Moreover, CTF defines discursive characteristics of Mathematics as the *word use*, *visual mediators*, *narratives*, and *routines* with their associated metarules, namely the *how* and the *when* of the routine. In addition, it involves the various objects of mathematical discourse such as the *signifiers*, *realisation trees*, *realisations*, *primary objects* and *discursive objects*. It also involves the constructs of *object-level* and *metadiscursive level* (or metalevel) *rules*. Thinking "is an individualized version of (interpersonal) communicating" (Sfard, 2008, p81). Contrary to the acquisitionist approaches, participationists' ontological tenets propose to consider thinking as an act (not necessarily interpersonal) of communication, rather than a step primary to communication (Nardi et al. 2014).

Mathematical discourse involves certain objects of different categories and characteristics. *Primary object* (p-object) is defined as "any perceptually accessible entity existing independently of human discourses, and this includes the things we can see and touch (material objects, pictures) as well as those that can only be heard (sounds)" (Sfard, 2008, p169). *Simple discursive objects* (simple d-objects) "arise in the process of proper naming (baptizing): assigning a noun or other noun-like symbolic artefact to a specific primary object. In this process, a pair <noun or pronoun, specific primary object> is created. The first element of the pair, the signifier, can now be used in communication about the other object in the pair, which counts as the signifier's only realization. *Compound discursive objects* (d-objects) arise by "according a noun or pronoun to extant objects, either discursive or primary." In the context of this study, groups are an example of compound d-objects.

Sfard (2008) describes two distinct categories of learning, namely the *object-level* and the *metalevel learning*. "Object-level learning […] expresses itself in the expansion of the existing discourse attained through extending a vocabulary, constructing new routines, and producing new endorsed narratives; this learning, therefore results in endogenous expansion of the discourse" (Sfard, 2008, p253). In addition, "metalevel learning, which involves changes in metarules of the discourse […] is usually related to exogenous change in discourse. This change means that some familiar tasks, such as, say, defining a word or identifying geometric figures, will now be done in a different, unfamiliar way and that certain familiar words will change their uses" (Sfard, 2008, p254).

Literature Review

A distinctive characteristic of advanced Mathematics in the university is the production of rigorous and consistent proofs. The often arduous, for the majority of students (Moore, 1994; Segal, 2000), task of successfully producing and communicating their proofs is a significant obstacle in the smooth transition from secondary to university Mathematics. Proof production is far from a straightforward task to analyse and identify the difficulties students face. From a pedagogical perspective, a possible contributing factor to the students' difficulty with proof is the teaching they receive both in high schools and in universities, since "most students have not been enculturated into the practice of proving, or even justifying the mathematical processes they use" (Dreyfus, 1999, p94). In addition, from a communicational perspective, there is a chasm between the professional mathematicians and students regarding their views about issues like conviction or validity of proof (Segal, 2000; Harel and Sowder, 1998), the adequacy of an explanation or justification (Sierpinska, 1994), or the ability to distinguish between different forms of reasoning (Dreyfus, 1999). Teachers often do not aim to give their students the means to learn how to construct proofs and judge their validity. This is a task left to students (Dreyfus, 1999).

Difficulties with proof production have been extensively investigated for various levels of student expertise (from novice undergraduates to experienced doctoral students). Moore (1994) classifies novice students' difficulties with proving in three wide categories referring to: the mathematical language and notation as such; the concept understanding; and, getting started with the proof. This categorisation is in conformity with the CTF where one is able to examine the students' use of words, visual mediators, understanding of the definitions of the related notions and the related theorems and lemmas, as well as the routines with their applicability and closure conditions. Weber (2001) categorises student difficulties with proofs into two classes: the first is related to the students' difficulty to have an accurate and clear conception of what comprises a mathematical proof, and the second is related to students' difficulty to understand a mathematical proposition or a concept and therefore systematically misuse it. In his study, an examination of the performance of undergraduate (novice) and doctoral (expert) students in proof-production, Weber (2001) indicated three types of strategic knowledge that the latter applied and the former lacked, in particular, referring to the knowledge of domain's proof techniques, the knowledge of which theorems are important and when they will be useful and the knowledge of when and when not to use 'syntactic' strategies.

Particularly difficult becomes the process of proof in the context of Group Theory, which is often considered one of the most difficult subjects for novice undergraduate Mathematics students (Ioannou, 2012, 2018). Several research studies within the last three decades have focused on the learning of Group Theory and on various aspects of students' learning experience, adopting various theoretical perspectives. For instance studies such as Dubinsky et al. (1994), Brown et al (1997) and Asiala et al (1996, 1997), which follow a constructivist approach, and within the Piagetian tradition of studying the cognitive processes, have examined students' cognitive development and analysed the emerging difficulties in the process of learning certain group-theoretic notions. Novice students are required to successfully cope with its abstract and rigorous nature and invent new learning approaches.

The abstract nature of Group Theory is, more often than not, an impediment for novice students. In order to successfully cope with learning Group Theory, students often tend to adopt techniques that would reduce the level of abstraction, which according to Hazzan (1999, p73) is an "effective mental strategy, which enables students to mentally cope with the new, abstract kind of mathematical objects". A significant number of students seem to have the tendency to work on a lower level of abstraction than the one in which concepts are introduced. By reducing the level of abstraction, students are enabled to "base their understanding on their current knowledge, and proceed towards mental construction of mathematical concepts conceived on higher level of abstraction" (Hazzan, 1999, p84). This is particularly obvious, when their first course in Group Theory progresses beyond the definition of the basic notions of group and subgroup (Ioannou, 2012).

Methodology

This study is a ramification of a larger study that focused on Year 2 Mathematics students' conceptual difficulties and the emerging learning and communicational aspects in their first encounter with Group Theory. The module was taught in a research-intensive Mathematics department in the United Kingdom.

The Abstract Algebra (Group Theory and Ring Theory) module was mandatory for Year 2 Mathematics undergraduate students, and a total of 78 students attended it. The module was spread over 10 weeks, with 20 one-hour lectures and three cycles of seminars in weeks 3, 6 and 10 of the semester. The role of the seminars was mainly to support the students with their coursework. There were 4 seminar groups, and the sessions were each facilitated by a seminar leader, a full-time faculty member of the school, and a seminar assistant, who was a PhD student in the Mathematics department. All members of the teaching team were pure mathematicians.

The lectures consisted largely of exposition by the lecturer, a very experienced pure mathematician, and there was not much interaction between the lecturer and the students. During the lecture, the lecturer wrote self-contained notes on the blackboard, while commenting orally at the same time. Usually, he wrote on the blackboard without looking at his handwritten notes.

In the seminars, the students were supposed to work on problem sheets, which were usually distributed to the students a week before the seminars. The students had the opportunity to ask the seminar leaders and assistants about anything they had a problem with and to receive help. The module assessment was predominantly exam-based (80%). In addition, the students had to hand in a threefold piece of coursework (20%) by the end of Week 12.

The data gathered includes the following: Lecture observation field notes, lecture notes (notes of the lecturer as given on the blackboard), audio-recordings of the 20 lectures, audio-recordings of the 21 seminars, 39 student interviews (13 volunteers who gave 3 interviews each), 15 members of staff's interviews (5 members of staff, namely the lecturer, two seminar leaders and two seminar assistants, who gave 3 interviews each), student coursework, markers' comments on student coursework, and student examination scripts. For the purposes of this study, there have been used data from the interviews.

Moreover, there have been used qualitative methods of data analysis, since this approach is a dynamic, intuitive and creative process of inductive thinking, reasoning and theorising. Data analysis allowed the researcher to comprehend the context of data and refine the interpretations. The core objective of this process was to determine the codes, categories, relationships and assumptions related to the topic under study (Cohen et al. 2007). Below it follows a detailed factual description of how the data was analysed.

Lecture observation notes and Lecture notes: Lecture observation notes were handwritten during the data collection process alongside the lecture note taking. Both of these data categories were summarised in 'Lecture Summaries'. In these summaries, the researcher organised the content and systematized the presentation of lecture observation notes and included a brief index of content for each lecture. At the end of each lecture summary, there followed a commentary on issues of content, presentation, general ambiance and other incidents of interest, and were connected with the prospective focus of this study, such as communication or any form of interaction between the lecturer and the students

Interviews: Both student and staff interviews were fully transcribed. Also they were illustrated with comments regarding the mood, voice tone, emotions and attitudes, or incidents of laughter, long pauses etc. The final interview documents were called Annotated Interview Transcriptions, since they were an annotated version of the interview transcription. In these annotated versions of interview transcriptions, the researcher highlighted certain phrases or even parts of the dialogues that were related to a particular theme.

Coursework and Examination Scripts: In the data analysis process, both the coursework solutions and the exam scripts were analysed last. Coursework student solutions were analysed in detail, mostly focusing on issues such as the use of certain concepts, the use of mathematical vocabulary and symbolisation, the use of language and the style of language used, the proof production process and the use of visual imagery and external objects. Concerning the examination scripts, the same approach was followed. In parallel, the comments of the markers were also analysed. Moreover, students' solutions were scrutinised, identifying related incidences and therefore, scanning the bits that were of interest. In doing so, the researcher produced data analysis documents, by scanning all the excerpts relevant to the purposes of this study, and analysing them using CTF.

Finally, all emerging ethical issues during the data collection and analysis, namely, issues of power, equal opportunities for participation, right to withdraw, procedures of complain, confidentiality, anonymity, participant consent, and sensitive issues in interviews, have been addressed accordingly.

Data Analysis

An often-reported difficulty that seven out of the thirteen (7/13) students faced was how to start a proof, i.e. the initial step. For many students, getting help on how to start a proof is a main incentive for attending the seminars, as the following excerpt suggests.

> […] *Most of the time I can't do them, and I need a little... sort of, starting point, so I – I – but I'm – I have a good kind of understanding about what the question's asking me to do, um, before I go, so I can ask the right questions... cos it's a lot more abstract, like you don't have anything to compare it to… I mean there's a lot of imaginary stuff in it…It's like nothing you've ever experienced before* […] *I mean I've done groups before, so I know kind of – the basics of it umm, just to get a greater understanding really.* Dorabella

The above suggests that normativeness of metarules is not automatically established among all members of the mathematical community. Normativeness of metarules in a certain discursive context is objective, since it requires experience. Well-established metarules that would allow experienced mathematicians to successfully solve a mathematical task would not automatically be obvious to the majority of novice students.

Moreover, difficulty with starting a proof is possibly an indication of incomplete metalevel learning regarding the how of a routine as well as its applicability conditions. Application of metalevel rules governs the formulation and substantiation of the object-level rules. Incomplete object level learning may hinder the successful application of the involved metarules and the construction of the required proof, although this is not always the case.

As the above excerpt suggests, Dorabella, similarly to other students, seems to have neither fully developed a thorough metalevel learning of the related metarules nor does she have a clear perspective of what she wants to prove and how she is going to prove it. This is not an unexpected or rare event among novice students in university Mathematics in general (Moore, 1994), and in Group Theory in particular, since students need to achieve a transition towards a higher level of abstraction, even between different university Mathematics modules.

Moreover, the starting point of a proof is a particularly difficult step, especially when students have seen nothing similar to the proof under discussion. The following excerpt of Leonora highlights the importance of previous experience for the development of metalevel learning.

> *I don't know, I mean I think once I've done it, and been told – like – having – so I've got like an example basically, of how to do it, then – it will be in my mind, so it'll be hopefully, something I can keep repeating, but just initially starting it off and – it's – I find quite hard, I found that quite hard with like a lot of things, it's just initially start…* Leonora

The above excerpt possibly highlights the need of some students to be guided in the first steps of the learning of a new mathematical discourse of guidance and examples. For these students, examples possibly have a twofold role, first to improve their object-level learning regarding the involved d-objects, and second to enrich their experience of how metarules should be applied.

The primary importance of the starting point in proof is also in agreement with Norma's perception about proofs, considering it, as the major obstacle to be overcome. After overcoming this threshold, it is much easier both to gain perspective and to move on.

> *I do try and do something, but it – it may not be like a specific question, cos sometimes, err, I just need help in getting started, and then once I've started I'm ok, it's just – finding like the thing to do first…* Norma

This is again an example of problematic encounter with the applicability conditions in the process of proving, as possibly a result of lack of experience for the majority of novice students. Deciding about the how and the when of a particular routine, namely what course of action to follow and how to initiate and finish a proof, is an essential part of metalevel learning of a particular mathematical discourse.

Discussions with students have revealed their perception about their general approach to required object-level and metalevel learning. It was apparent in the case of five of the thirteen (5/13) students that instead of trying to grasp the related d-objects and the amenable metarules in proof production process, these students, at least at the initial stage of their learning, would excessively depend on similar examples, Internet and book use or other exogenous factors, and mechanically imitate them. This approach to proving, and learning in general is clearly expressed in the following excerpt.

> *I googled for one of the proofs, to see if it was on there, but it wasn't, but there was something similar, that then I worked out like – that you're just meant to go through and then times it by the inverses and stuff, but um – I think, some of our algebra nowadays is so... specific, that like there aren't proofs and stuff out there that's – that's easy to find now.* Amelia

Overdependence on exogenous sources for understanding instead of focusing on the endogenous change of discourse is obvious in the above excerpt. Apparently, some students need to see similar routines to the ones they have to produce in the context of an exercise. Studying similar routines possibly helps them to grasp the metarules that need to be applied in certain situations and to be able, at the first stages of their learning, to imitate. In addition, Amelia seems to realise the level of specificity and rigour that proofs in Abstract Algebra require through her effort to find similar proofs. Nevertheless she uses the 'similar' proof to her benefit in order to learn the metarules related to notions such as inverses (see Ioannou, 2012).

Regarding the issue of self-evaluation of students' produced proofs, many students, based on their experience so far, show an awareness of the quality of their proof as well as whether their proofs are correct or not, even when they closely follow a routine similar to the one they are supposed to produce. In the excerpt below, Francesca discusses the importance of a logical order of steps of a proof, without missing any.

> *You should follow the correct order though… because I remember once I was jumping steps in a proof and I was considering some steps as granted… I shouldn't though… I had to prove every step and then go to the next step… That's why I lose so many points… because I arrive at the correct result but I was missing some things…* Francesca

She seems to be aware of the normative nature of metarules and that she is expected to meet the required characteristics of explicitness and rigour. She realises that proof production is in a way a very particular activity within a certain discourse, which has well-established metarules. Even though she is willing to apply a routine that will lead her to an endorsable and valid mathematical narrative, she does not yet possess the required capability to apply the involved metarules that will allow her to achieve this, in this specific mathematical discourse.

This claim is in accordance with the analysis in Ioannou (2012), where in many occasions, students' reported intention of action is in disagreement with the action taken, even if they are aware of the inappropriateness of their actions.

Students' own evaluation of their proofs is an interesting issue from the secondary-tertiary level Mathematics transition. According to Gueudet (2008), novice students do not have the experience that will allow them to decide whether a proof is valid or not. Nevertheless the above excerpt and the ones that follow shows that students, although they may not have the capability to precisely evaluate their proofs, can say however, whether it is problematic or inadequate, based on their previous experience.

A representative example of contradiction between the reported intention of proof strategy and the applied action can be seen in the following excerpt. Manrico, although he eventually does so in practice, is nevertheless aware that a proof, which is over-dependent on visual mediators, is neither rigorous enough nor acceptable in the context of Group Theory, and consequently not in accordance with the metalevel norms.

> *Hmm. see, there – this thing – I mean – the – statement, makes sense... I drew a little picture and like – I was just like – I mean – course that's going to be in it, but – how you prove that by actual kind of – prove that mathematically rather than just drawing a picture and just saying, it is true, it's just the actual showing that...* Manrico

Even though the use of visual mediators is an important aspect of mathematical discourse and often an indication of good object-level learning, extensive use is not an adequate approach for proving a mathematical narrative, especially in the context of Group Theory, at least as it is taught in the specific Mathematics department. Excessive use of illustrations and lack of algebraic reasoning in the proofs often indicates incomplete metalevel learning.

Proof production also depends on the thorough learning of d-objects and their realization trees, as well as the good interaction between the object-level and metalevel rules. The excerpt below reveals the negative consequences on the interaction between the different realisation trees as these have been developed possibly in different modules.

> *I cannot understand many things that… for example… one of the things I cannot accept is… I am given an exercise in which I have to prove something and in the notes we are not given something that will help us or guide us to solve the exercise… or the fact that something that we see now it is related to something that we have seen several months ago…* Musetta

The above suggests that Musetta's learning lacks connectivity between the different modules in her degree. Although she is aware of this lack, her approach towards overcoming this issue is rather passive, and as her written data suggests (see Ioannou, 2012), she has not effectively achieved it. Moreover, if the d-objects and the corresponding realization trees involved in a particular discourse or other related ones have not been encapsulated, then discursive expansion has yet to be achieved. If the student is not able to construct usable and accessible realization trees, then proof production is very difficult, if not impossible to be achieved. As Ioannou (2012) suggests, mathematical learning in Group Theory requires realization trees that involve compound d-objects emerging by reification, namely regarding the shift of the focus from processes on the group-theoretic objects towards discussions of the group-theoretic objects as such and their relations.

Conclusion

This study has focused on novice undergraduate Mathematics students' difficulties with proof production in the context of Abstract Algebra. For the majority of these students, proof is an arduous task. In an abstract mathematical discourse such as Abstract Algebra, students are invited to produce proofs for several mathematical problems, both in the coursework and in the examination. As the literature (Moore, 1994; Harel and Sowder, 1998; Weber, 2001) and the discussion above suggests, proof production, especially in Group Theory, represents a particular challenge, because students have to develop several indispensable skills, such as their ability to cope with the abstract nature of this module and a certain flexibility in the application of metarules. It cannot be assumed that the majority of the students can develop these skills instantly or easily.

In addition, proof production is a new element in the students' learning experience, requiring successful application of both object-level and metalevel rules, and therefore a challenge in the secondary-tertiary transition that needs to be confronted. As the discussion above suggests, many novice students often have difficulty with the 'how' and 'when' of the required routines. Successful proof production depends on the thorough object-level learning of involved d-objects and their realization trees, as well as the successful and precise application of the governing metalevel rules, in the particular context. Evidently, students face various difficulties with the three steps of the procedure for developing a certain routine, namely, the applicability conditions, the course of action and the closure conditions. In particular, some students often face difficulties initiating a proof. It is a difficulty that frequently occurred in the context of this study and was also identified by Moore (1994). In addition, some students often have difficulty in recognising the signs that would signal the end of the proof, leaving them with a feeling of doubt. A future study will investigate students' perceptions about mathematical proof, as a way of mathematical communication.

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